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On the structure of measurable filters on a countable set

Abstract

A combinatorial characterization of measurable filters on a countable set is found. We apply it to the problem of measurability of the intersection of nonmeasurable filters.

The goal of this paper is to characterize measurable filters on the set of natural numbers. In section 1 we introduce basic notions, in section 2 we find a combinatorial characterization of measurable filters, in section 3 we study intersections of filters and finally section 4 is devoted to filters which are both null and meager.

Through this paper we use standard notation. ω denotes the set of natural numbers. For $k, n \in \omega$ let $[n, k] = \{i < \omega : n \leq i \leq k\}$. For $n \in \omega$, 2^n (2^ω) denotes the set of 0-1 sequences of length $n(\omega)$, also let $2^{<\omega} = \bigcup_{n\in\omega} 2^n$. For any sequences $s, t \in 2^{<\omega}$ let $s^\frown t$ denote their concatenation. For $s \in 2^{<\omega}$ let $[s] = \{x \in 2^\omega : s \subset x\}$. The family $\{[s] : s \in 2^{<\omega}\}$ is a base of the space 2^ω. We will often identify a set $[s]$ with a sequence s and we will also identify subsets of ω with their characteristic functions. Filters considered in this paper are assumed to be nonprincipal. We identify filters on ω with sets of characteristic functions of its elements. In this way the question about measurability makes sense. Finally let quantifiers "$\exists^\infty$" and "$\forall^\infty$" denote "for infinitely many" and "for all except finitely many" respectively.

1 Introduction

In this section we establish some definitions and recall several facts which we will use later. Let us start with measures we will be working with.

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DEFINITION Let $\hat{p} = \{p_n : n \in \omega\}$ be a sequence of reals such that $p_n \in (0, \frac{1}{2}]$ for all $n \in \omega$. Define $\mu_{\hat{p}}$ to be the product measure on 2^ω such that $\mu_{\hat{p}}(\{x \in 2^\omega : x(n) = 1\}) = p_n$ and $\mu_{\hat{p}}(\{x \in 2^\omega : x(n) = 0\}) = 1 - p_n$ for $n \in \omega$. Notice that if $p_n = \frac{1}{2}$ then $\mu_{\hat{p}}$ is the usual measure on 2^ω. From now on let us fix one of the measures $\mu_{\hat{p}}$. We have the following:

Theorem 1.1 (Sierpinski) *Suppose that $\mathcal{F}$ is a filter on ω. Then $\mathcal{F}$ is either of $\mu_{\hat{p}}$-measure zero or $\mathcal{F}$ is $\mu_{\hat{p}}$-nonmeasurable. Moreover, $\mathcal{F}$ is either meager or does not have the Baire property.*

PROOF: In the case of the Baire property or when $\mu_{\hat{p}}$ is the Lebesgue measure we use the fact that the automorphism of 2^ω which sends every set to its complement preserves Lebesgue measure.

Suppose that a filter $\mathcal{F}$ is $\mu_{\hat{p}}$-measurable. Since $\mathcal{F}$ is non-principal it has measure 0 or 1. We have to show that $\mu_{\hat{p}}(\mathcal{F}) = 0$.

Consider $\varphi : 2^\omega \times 2^\omega \longrightarrow 2^\omega$ defined as $\varphi(X, Y)(n) = \max(X(n), Y(n))$ for $X, Y \in 2^\omega$.

Let q_n be chosen in such a way that $(1 - p_n)(1 - q_n) = 1/2$ for all n.

Claim 1.2 $\varphi^{-1}(\mu) = \mu_{\hat{p}} \times \mu_{\hat{q}}$.

PROOF: Verify that for all $n \in \omega$,

$$\frac{1}{2} = \mu(\{x \in 2^\omega : x(n) = 0\}) = \mu_{\hat{p}} \times \mu_{\hat{q}}(\varphi^{-1}(\{x \in 2^\omega : x(n) = 0\}))$$

and use the fact that sets of this form are independent. ■

Since φ is a continuous mapping it follows that if $A \subseteq 2^\omega \times 2^\omega$ and $\mu_{\hat{p}} \times \mu_{\hat{q}}(A) = 1$ then $\mu(\varphi(A)) = 1$.

Consider the set $\mathcal{F} \times 2^\omega$. If $\mu_{\hat{p}}(\mathcal{F}) = 1$ then $\mu_{\hat{p}} \times \mu_{\hat{q}}(\mathcal{F} \times 2^\omega) = 1$. In particular $\mu(\varphi(\mathcal{F} \times 2^\omega)) = \mu(\mathcal{F}) = 1$. Contradiction. ■

Sierpinski also proved that if $\mathcal{F}$ is an ultrafilter then $\mathcal{F}$ is Lebesgue nonmeasurable. The next theorem shows that with measures $\mu_{\hat{p}}$ this is not the case.

Theorem 1.3 *Let $\mathcal{F}$ be a filter on ω.*

1. *If there exists $X \in \mathcal{F}$ such that $\sum_{n \in X} p_n^k < \infty$ for some $k \in \omega$ then $\mu_{\hat{p}}(\mathcal{F}) = 0$.*

2. *Let $\{k_n : n \in \omega\}$ be a sequence of natural numbers such that*

$$\sum_{n=1}^{\infty} p_n^{k_n} = \infty \text{ but } \sum_{n=1}^{\infty} p_n^{k_n+1} < \infty.$$

If for every $X \in \mathcal{F}$, $\sum_{n\in X} p_n^{k_n} = \infty$ then $\mu_{\hat{p}}(\mathcal{F}) = 0$.

PROOF: The first part of this theorem is due to M. Talagrand. For completeness we sketch the proofs of both parts.

1) If there exists $X \in \mathcal{F}$ such that $\sum_{n\in X} p_n < \infty$ then $\mu_{\hat{p}}(\mathcal{F}) = 0$ since $\mu_{\hat{p}}(\{Y \subseteq \omega : |X \cap Y| = \omega\}) = 0$. If there exists $X \in \mathcal{F}$ such that $\sum_{n\in X} p_n^{k+1} < \infty$ but for all $Y \in \mathcal{F}$, $\sum_{n\in Y} p_n^k = \infty$ then $\mu_{\hat{p}}(\mathcal{F}) = 0$ since $\mu_{\hat{p}}(\{Y \subseteq \omega : |Y \cap X| = \omega \text{ and } \sum_{n\in X\cap Y} p_n^k = \infty\}) = 0$.

2) Consider the random variables

$$\xi_n(X) = \begin{cases} p_n^{k_n} & \text{if } n \in X \\ 0 & \text{otherwise} \end{cases}.$$

Let $\xi = \sum_{n=1}^{\infty} \xi_n$. The mean value $E(\xi) = \sum_{n=1}^{\infty} p_n^{k_n+1} < \infty$. Therefore $\mu_{\hat{p}}(\{X \subseteq \omega : \sum_{n\in X} p_n^{k_n} = \infty\}) = 0$. ∎

The next theorem characterizes filters having the Baire property.

Theorem 1.4 (Talagrand [T1]) *For any filter $\mathcal{F}$ on ω the following conditions are equivalent:*

1. *$\mathcal{F}$ has the Baire property,*
2. *there exists a partition of ω $\{I_n : n \in \omega\}$ into finite intervals such that $\forall X \in \mathcal{F}\ \forall^\infty n\ X \cap I_n \neq \emptyset$.* ∎

Our first goal is to describe certain family of $\mu_{\hat{p}}$-null sets which will be used to cover $\mu_{\hat{p}}$-measurable filters.

DEFINITION Set $H \subseteq 2^\omega$ is called *small* with respect to the measure $\mu_{\hat{p}}$ if there exists a partition of ω into pairwise disjoint intervals $\{I_n : n \in \omega\}$ and a sequence $\{J_n : n \in \omega\}$ such that

1. $J_n \subseteq 2^{I_n}$ for $n \in \omega$,

2. $H \subseteq \{x \in 2^\omega : \exists^\infty n \; x \restriction I_n \in J_n\}$,

3. $\mu_{\hat{p}}(\{x \in 2^\omega : \exists^\infty n \; x \restriction I_n \in J_n\}) = 0$.

Denote the set occuring in 2) and 3) by $(I_n, J_n)_{n=1}^\infty$. Notice that by Borel-Cantelli lemma we can replace condition (3) by the equivalent :

$$3'. \quad \sum_{n=1}^{\infty} \mu_{\hat{p}}(\{x \in 2^\omega : x \restriction I_n \in J_n\}) < \infty.$$

The following generalizes the theorem from [Ba].

Theorem 1.5 *1. Every $\mu_{\hat{p}}$-null set is a union of two $\mu_{\hat{p}}$-small sets,*

2. There exists $\mu_{\hat{p}}$-null set which is not small.

Proof: For completeness we sketch the proof of the first part. Fix $\mu_{\hat{p}}$ for some $\hat{p} = \{p_n : n \in \omega\}$ and let $H \subseteq 2^\omega$ be a $\mu_{\hat{p}}$-null set. The following claim is implicit in [O].

Claim 1.6 *$\mu_{\hat{p}}(H) = 0$ iff there exists a sequence $\{F_n \subseteq 2^n : n \in \omega\}$ such that*

1. $H \subseteq \{x \in 2^\omega : \exists^\infty n \; x \restriction n \in F_n\}$,

2. $\sum_{n=1}^\infty \mu_{\hat{p}}(\{x \in 2^\omega : x \restriction n \in F_n\}) < \infty$.

Proof: The only difference between this and the definition of a small set is that "domains" of F_n's are not disjoint.

$\leftarrow$ This implication is an immediate consequence of Borel-Cantelli lemma.

$\rightarrow$ Since $\mu_{\hat{p}}(H) = 0$ there are open sets $\{G_n : n \in \omega\}$ covering H such that $\mu_{\hat{p}}(G_n) < \frac{1}{2^n}$ for $n \in \omega$. Write each G_n as a union of disjoint basic sets i.e.

$$G_n = \bigcup_{m \in \omega} [s_m^n] \text{ for } n \in \omega.$$

Let $F_n = \{s \in 2^n : s = s_k^l \text{ for some } k, l \in \omega\}$ for $n \in \omega$. Verification of 1) and 2) is straightforward. ■

Using the above claim and the assumption that $\mu_{\hat{p}}(H) = 0$ we can find a sequence $\{F_n : n \in \omega\}$ satisfying the conditions 1) and 2) of the claim. Fix

a sequence of positive reals $\{\varepsilon_n : n \in \omega\}$ such that $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ and let $q(n) = p_1^{-1} \cdot p_2^{-1} \cdot \ldots \cdot p_n^{-1}$ for $n \in \omega$.

Define two sequences $\{n_k, m_k : k \in \omega\}$ as follows: $n_0 = 0$,

$$m_{k+1} = \min\{u > n_k : q(n_k) \cdot \sum_{j=u}^{\infty} \mu_{\hat{p}}(\{x \in 2^\omega : x \restriction j \in F_j\}) < \varepsilon_k\},$$

$$n_{k+1} = \min\{u > m_{k+1} : q(m_{k+1}) \cdot \sum_{j=u}^{\infty} \mu_{\hat{p}}(\{x \in 2^\omega : x \restriction j \in F_j\}) < \varepsilon_k\},$$

and let

$$I_k = [n_k, n_{k+1}) \text{ and } I'_k = [m_k, m_{k+1}) \text{ for } k \in \omega.$$

Let

$$s \in J_k \text{ iff } s \in 2^{I_k} \text{ and } \exists i \in [m_{k+1}, n_{k+1})\ \exists t \in F_i\ s \restriction \mathrm{dom}(t) \cap \mathrm{dom}(s) =$$

$$t \restriction \mathrm{dom}(t) \cap \mathrm{dom}(s).$$

Similarly

$$s \in J'_k \text{ iff } s \in 2^{I'_k} \text{ and } \exists i \in [n_k, m_{k+1})\ \exists t \in F_i\ s \restriction \mathrm{dom}(t) \cap \mathrm{dom}(s) =$$

$$t \restriction \mathrm{dom}(t) \cap \mathrm{dom}(s).$$

It remains to show that $(I_k, J_k)_{k=1}^{\infty}$ and $(I'_k, J'_k)_{k=1}^{\infty}$ are small with respect to $\mu_{\hat{p}}$ and that their union covers H. Consider set $(I_k, J_k)_{k=1}^{\infty}$. Notice that

$$\mu_{\hat{p}}(\{x \in 2^\omega : x \restriction I_k \in J_k\}) \leq$$

$$\leq \mu_{\hat{p}}(\{x \in 2^\omega : \exists i \in [m_{k+1}, n_{k+1})\ \exists t \in F_i\ x \restriction [m_{k+1}, i) = t \restriction [m_{k+1}, i)\}) \leq$$

$$q(n_k) \cdot \sum_{i=m_{k+1}}^{n_{k+1}-1} \mu_{\hat{p}}(\{x \in 2^\omega : \exists t \in F_i\ x \restriction i = t\}) \leq \varepsilon_k.$$

Since $\sum_{n=1}^{\infty} \varepsilon_n < \infty$ this shows that the set $(I_n, J_n)_{n=1}^{\infty}$ is $\mu_{\hat{p}}$-null.

Analogous argument works for the other set. Finally we have that

$$H \subseteq (I_n, J_n)_{n=1}^{\infty} \cup (I'_n, J'_n)_{n=1}^{\infty}.$$

To see this suppose that $x \in H$. Then the set $X = \{n \in \omega : x \restriction n \in F_n\}$ is infinite. Thus either

$$X \cap \bigcup_{k=1}^{\infty} [m_{k+1}, n_{k+1}) \text{ is infinite or}$$

$$X \cap \bigcup_{k=1}^{\infty} [n_k, m_{k+1}) \text{ is infinite .}$$

Without loss of generality we can assume that it is the first case. But it means that $x \in (I_n, J_n)_{n=1}^{\infty}$ because if $x \restriction n \in F_n$ and $n \in [m_{k+1}, n_{k+1})$ then by the definition of J_k there is $t \in J_k$ such that $x \restriction [n_k, n_{k+1}) = t$. We are done since it happens infinitely many times. ■

2 Measurable filters

In this section we characterize $\mu_{\hat{p}}$-measurable filters on ω.

Theorem 2.1 *Let $\mathcal{F}$ be a filter on ω. Then $\mathcal{F}$ is $\mu_{\hat{p}}$-measurable iff $\mathcal{F}$ is $\mu_{\hat{p}}$-small.*

PROOF: $\leftarrow$ This implication is obvious.

$\rightarrow$ Let $\mathcal{F}$ be a $\mu_{\hat{p}}$-measurable filter. Fix a sequence $\{\varepsilon_n : n \in \omega\}$ of positive reals such that $\sum_{n=1}^{\infty} 2^n \cdot \varepsilon_n < \infty$.

For $J \subseteq 2^{<\omega}$ let $V(J) = \{x \in 2^{\omega} : \exists s \in J \; s \subset x\}$.

By 1.1 we know that $\mathcal{F}$ can be covered by some $\mu_{\hat{p}}$-null set $H \subseteq 2^{\omega}$. Applying 1.5 we can find two $\mu_{\hat{p}}$-small sets covering H. In fact as in the proof of 1.5 we can find sequences $\{n_k, m_k : k \in \omega\}$ and families $\{J_k, J'_k : k \in \omega\}$ such that

1. $n_k < m_{k+1} < n_{k+1} < m_{k+2}$ for $k \in \omega$,
2. $J_k \subseteq 2^{[n_k, n_{k+1})}$, $J'_k \subseteq 2^{[m_k, m_{k+1})}$ for $k \in \omega$,
3. $\mu_{\hat{p}}(V(J_k)) < \varepsilon_k$, $\mu_{\hat{p}}(V(J'_k)) < \varepsilon_k$ for $k \in \omega$,
4. $H \subseteq ([n_k, n_{k+1}), J_k)_{k=1}^{\infty} \cup ([m_k, m_{k+1}), J'_k)_{k=1}^{\infty}$.

Now for $k \in \omega$ define

$$S_k = \left\{ t \in 2^{[n_k, m_{k+1})} : \mu_{\hat{p}}(V(\{s \in 2^{[m_{k+1}, n_{k+1})} : t^\frown s \in J_k\})) > \frac{1}{2^k} \right\}.$$

Notice that for all $k \in \omega$

$$\frac{1}{2^k} \mu_{\hat{p}}(V(S_k)) \leq \mu_{\hat{p}}(J_k) < \varepsilon_k \text{ hence}$$

$$\mu_{\hat{p}}(V(S_k)) \leq 2^k \cdot \varepsilon_k.$$

Similarly if for $k > 1$ we define

$$S'_k = \left\{ t \in 2^{[n_k, m_{k+1})} : \mu_{\hat{p}}(V(\{s \in 2^{[m_k, n_k)} : s^\frown t \in J'_k\})) > \frac{1}{2^k} \right\}$$

then $\mu_{\hat{p}}(V(S'_k)) \leq 2^k \cdot \varepsilon_k$.

Thus we have three $\mu_{\hat{p}}$-small sets

$$H_1 = ([n_k, n_{k+1}), J_k)_{k=1}^\infty,$$

$$H_2 = ([m_k, m_{k+1}), J'_k)_{k=1}^\infty \text{ and}$$

$$H_3 = ([n_k, m_k), S_k \cup S'_k)_{k=1}^\infty.$$

If $\mathcal{F} \subseteq H_2 \cup H_3$ we are done since it is easy to see that it is a $\mu_{\hat{p}}$-small set. So assume that there exists $X \in \mathcal{F}$ such that $X \in H_1$ but $X \notin H_2 \cup H_3$. Since $X \in H_1$ we have an infinite sequence $\{k_u : u \in \omega\}$ such that

$$\forall u \in \omega \; X \restriction [n_{k_u}, n_{k_u+1}) \in J_{k_u}.$$

Define for $u \in \omega$

$$U_u = [m_{k_u+1}, n_{k_u+1}) \text{ and}$$

$$T_u = \{s \in 2^{U_u} : X \restriction [n_{k_u}, m_{k_u+1})^\frown s \in J_{k_u} \textit{ or } s^\frown X \restriction [n_{k_u+1}, m_{ku+2}) \in J'_{k_u+1}\}.$$

We have to check that the set $(U_u, T_u)_{u=1}^\infty$ is $\mu_{\hat{p}}$-small. Consider sufficiently large $u \in \omega$. Since $X \restriction [n_{k_u}, n_{k_u+1}) \in J_{k_u}$ and $X \restriction [n_{k_u}, m_{k_u+1}) \notin S_{k_u} \cup S'_{k_u}$ (u is large) we have $\mu_{\hat{p}}(V(T_u)) < 2^{-u}$.

Claim 2.2 $\mathcal{F} \subseteq (U_u, T_u)_{u=1}^\infty$.

Proof: Suppose that $\mathcal{F}$ is not contained in this set and let $Y \in \mathcal{F} - (U_u, T_u)_{u=1}^{\infty}$. Define $Z \in 2^{\omega}$ as follows

$$Z(n) = \begin{cases} X(n) & \text{if } n \in \bigcup_{u\in\omega} U_u \\ Y(n) & \text{otherwise} \end{cases} \quad \text{for } n \in \omega.$$

Notice that $Z \in \mathcal{F}$ since $X \cap Y \subseteq Z$. We will show that $Z \notin H_1 \cup H_2$ which gives a contradiction. Consider an interval $I_m = [n_m, n_{m+1})$. If $m \neq k_u$ for every $u \in \omega$ then $I_m \cap \bigcup_{u\in\omega} U_u = \emptyset$ and $Z \restriction I_m \notin J_m$ since $Z \restriction I_m = X \restriction I_m$ for such m's. On the other hand if $m = k_u$ for some $u \in \omega$ then $X \restriction I_m \in J_m$ but by the choice of X $Z \restriction [n_{k_u}, m_{k_u+1}) = X \restriction [n_{k_u}, m_{k_u+1})$ has only few extensions inside J_{k_u} (since $X \notin H_3$). In fact if $Z \restriction I_m \in J_m$ then $Z \restriction [m_{k_u+1}, n_{k_u+1})$ has to be an element of T_u. But this is impossible since $Z \restriction [m_{k_u+1}, n_{k_u+1}) = Y \restriction [m_{k_u+1}, n_{k_u+1}) \notin T_u$ for sufficiently large $u \in \omega$. Hence for all except finitely many $m \in \omega$ $Z \restriction I_m \notin J_m$ which means that $Z \notin H_1$. Similarly, using the second clause in the definition of H_3 we prove that $Z \notin H_2$. That finishes the proof since $(U_u, T_u)_{u=1}^{\infty}$ is a $\mu_{\hat{p}}$-small set. ■

As a corollary we get:

Theorem 2.3 *For any filter $\mathcal{F}$ the following conditions are equivalent:*

1. *$\mathcal{F}$ is $\mu_{\hat{p}}$-measurable,*
2. *there exists a family $\{\mathcal{A}_n : n \in \omega\}$ such that*
   - (a) *$\mathcal{A}_n$ consists of finitely many finite subsets of ω for all $n \in \omega$,*
   - (b) *$\bigcup \mathcal{A}_n \cap \bigcup \mathcal{A}_m = \emptyset$ whenever $n \neq m$,*
   - (c) *$\sum_{n=1}^{\infty} \mu_{\hat{p}}(\{X \subseteq \omega : \exists a \in \mathcal{A}_n\ a \subset X\}) < \infty$,*
   - (d) *$\forall X \in \mathcal{F}\ \exists^{\infty} n\ \exists a \in \mathcal{A}_n\ a \subset X$.*

Proof: 2) $\rightarrow$ 1) This implication is obvious.

1) $\rightarrow$ 2). Assume that $\mathcal{F}$ is a measurable filter. Then by the previous theorem $\mathcal{F} \subset (I_n, J_n)_{n=1}^{\infty}$ for some $\mu_{\hat{p}}$-small set $(I_n, J_n)_{n=1}^{\infty}$. Define for $n \in \omega$

$$J'_n = \{s \in J_n : \forall u \in 2^{I_n}\ (s^{-1}(1) \subseteq u^{-1}(1) \rightarrow u \in J_n\}.$$

Claim 2.4 $\mathcal{F} \subseteq (I_n, J'_n)_{n=1}^{\infty}$.

PROOF: Suppose not. Let $X \in \mathcal{F} - (I_n, J'_n)_{n=1}^{\infty}$. It is not very hard to see that there exists a set $X' \supseteq X$ which does not belong to $(I_n, J_n)_{n=1}^{\infty}$. Contradiction. ■

Identify elements of J'_n with subsets of I_n and let

$$\mathcal{A}_n = \{a \subseteq I_n : a \text{ is } \subseteq -\text{minimal element of } J'_n\} \text{ for } n \in \omega.$$

Obviously $\mathcal{F} \subseteq \{X \subseteq \omega : \exists^{\infty} n \ \exists a \in \mathcal{A}_n \ a \subset X\}$ and the family $\{\mathcal{A}_n : n \in \omega\}$ has properties $a) - d)$. ■

If a family $\{\mathcal{A}_n : n \in \omega\}$ has the properties $a) - c)$ denote the set $\{X \subseteq \omega : \exists^{\infty} n \ \exists a \in \mathcal{A}_n \ a \subset X\}$ by $(\mathcal{A}_n)_{n=1}^{\infty}$. Characterization proved above can be interpreted as follows: Filter $\mathcal{F}$ is $\mu_{\hat{p}}$-null iff there exists a sequence of independent "tests" $\{\mathcal{A}_n : n \in \omega\}$ such that every element of $\mathcal{F}$ passes infinitely many of them. Condition (c) is a necessary requirement for such a set to have measure zero. Using 2.3 one can prove that

Theorem 2.5 *Let $\mathcal{F}$ be a filter on ω.*

1. *Let $\mu_{\hat{p}}$ and $\mu_{\hat{q}}$ be two measures such that $p_n \leq q_n$ for all except finitely many n. Then $\mu_{\hat{p}}(\mathcal{F}) = 0$ whenever $\mu_{\hat{q}}(\mathcal{F}) = 0$.*

2. *Let $\mu_{\hat{p}}$ and $\mu_{\hat{q}}$ be two measures such that $\mathcal{F}$ is nonmeasurable with respect to both of them. Define $r_n = \min\{p_n, q_n\}$ for $n \in \omega$. Then $\mathcal{F}$ is $\mu_{\hat{r}}$-nonmeasurable.*

PROOF: 1) This can be showed be direct computation. A more sophisticated but shorter is the following:

Suppose that $\hat{p}$ and $\hat{q}$ are two sequences such that $p_n \leq q_n$ for $n \in \omega$. It is enough to show that for any set $A = (\mathcal{A}_n)_{n=1}^{\infty}$, $\mu_{\hat{p}}(A) = 0$ whenever $\mu_{\hat{q}}(A) = 0$.

Let $\varphi : P(\omega) \times P(\omega) \longrightarrow P(\omega)$ be the mapping defined as $\varphi(X, Y) = X \cup Y$ for $X, Y \in P(\omega)$. Let

$$r_n = 1 - \frac{1 - q_n}{1 - p_n} \ for \ n \in \omega.$$

As in 1.2 we show that $\varphi^{-1}(\mu_{\hat{q}}) = \mu_{\hat{p}} \times \mu_{\hat{r}}$.

Since $\mu_{\hat{q}}(A) = 0$ we have that $\mu_{\hat{p}} \times \mu_{\hat{r}}(\varphi^{-1}(A)) = 0$. Therefore by Fubini's theorem there is set $X \subseteq \omega$ such that

$$\mu_{\hat{p}}(\{Y \subseteq \omega : X \cup Y \in (\mathcal{A}_n)_{n=1}^{\infty}\}) = 0.$$

But $\{Y \subseteq \omega : X \cup Y \in (\mathcal{A}_n)_{n=1}^{\infty}\} \supseteq (\mathcal{A}_n)_{n=1}^{\infty}$.

2) Suppose that $X \subseteq \omega$ is an infinite set. Call X, $\mathcal{F}$-positive if the family $\mathcal{F} \cup \{X\}$ generates a proper filter.

If X is $\mathcal{F}$-positive let

$$\mathcal{F}_X = \{X \cap Y : Y \in \mathcal{F}\}$$

be a trace of $\mathcal{F}$ on X. We will use the following fact:

Claim 2.6 *For every filter $\mathcal{F}$ and measure $\mu_{\hat{p}}$ the following conditions are equivalent.*

1. *$\mathcal{F}$ is $\mu_{\hat{p}}$-nonmeasurable,*
2. *$\mathcal{F}_X$ is $\mu_{\hat{p}\restriction X}$-nonmeasurable for every $\mathcal{F}$-positive set $X \subseteq \omega$,*
3. *there exists $\mathcal{F}$-positive set $X \subseteq \omega$ such that $\mathcal{F}_X$ is $\mu_{\hat{p}\restriction X}$-nonmeasurable.*

PROOF: 1) $\rightarrow$ 2) Suppose that $\mu_{\hat{p}\restriction X}(\mathcal{F}_X) = 0$ for some $\mathcal{F}$-positive set X. Then by 2.3 there exists $\mu_{\hat{p}\restriction X}$-small set $A = (\mathcal{A}_n)_{n=1}^{\infty} \subset 2^X$ which covers $\mathcal{F}_X$. Since A is upwards-closed it is easy to see that this set covers $\mathcal{F}$ as well and is $\mu_{\hat{p}}$-small.

2) $\rightarrow$ 3) Obvious.

3) $\rightarrow$ 1) Suppose that $\mathcal{F}_X$ is $\mu_{\hat{p}\restriction X}$-nonmeasurable for some $X \subseteq \omega$. Notice that

$$\mathcal{F}_X \times 2^{\omega - X} \subseteq \mathcal{F}.$$

That finishes the proof since $\mu_{\hat{p}} = \mu_{\hat{p}\restriction X} \times \mu_{\hat{p}\restriction \omega - X}$. ■

Now we can finish the proof of 2.5. Suppose that $\mathcal{F}$ is nonmeasurable with respect to measures $\mu_{\hat{p}}$ and $\mu_{\hat{q}}$. Let $r_n = \min\{p_n, q_n\}$ for $n \in \omega$. We show that $\mathcal{F}$ is $\mu_{\hat{r}}$-nonmeasurable. Let $X = \{n \in \omega : p_n = r_n\}$. Clearly either X or $\omega - X$ is $\mathcal{F}$-positive. Without loss of generality we can assume that we are in the first case. Applying the above claim and using the fact that $\mu_{\hat{p}\restriction X}$ and $\mu_{\hat{r}\restriction X}$ are the same measures on 2^X we get the desired conclusion. ■

Definition Filter $\mathcal{F}$ is called *rapid* if for every increasing function $f \in \omega^\omega$ there exists $X \in \mathcal{F}$ such that $|X \cap f(n)| \leq n$ for $n \in \omega$.

As another application we get a simple proof of the following result of Mokobodzki.

Theorem 2.7 (Mokobodzki) *Every rapid filter is Lebesque nonmeasurable.*

Proof: Let $\mathcal{F}$ be a rapid filter. Suppose that $\mathcal{F}$ is covered by a set of form $\{X \subset \omega : \exists^\infty n \; \exists a \in \mathcal{A}_n \; a \subset X\}$ where $\{\mathcal{A}_n : n \in \omega\}$ is a family as in 2.3. Without loss of generality we can assume that for all $n \in \omega$,

$$\mu(\{X \subseteq \omega : \exists a \in \mathcal{A}_n \; a \subset X\}) < \frac{1}{2^{n+1}}$$

and that

$$\max\{\max(a) : a \in \mathcal{A}_n\} \geq \min\{\min(a) : a \in \mathcal{A}_m\} \text{ for } n \geq m.$$

In particular it means that no set in $\mathcal{A}_n$ has less than $n+1$ elements. Define $f(n) = \max\{\max(a) : a \in \mathcal{A}_n\}$ for $n \in \omega$ and let $Z \in \mathcal{F}$ be such that $|Z \cap f(n)| \leq n$ for all $n \in \omega$. We immediately get that

$$Z \notin \{X \subset \omega : \exists^\infty n \; \exists a \in \mathcal{A}_n \; a \subset X\}.$$

Contradiction. ■

Before we go any further let us study the possible strenthening of 2.3. For simplicity we work with standard measure on 2^ω. Suppose that $(\mathcal{A}_n)_{n=1}^\infty$ is a small set. Notice that for given $n \in \omega$

$$\mu(\{X \subseteq \omega : \exists a \in \mathcal{A}_n \; a \subset X\}) \leq \sum_{a \in \mathcal{A}_n} \frac{1}{2^{|a|}} - \sum_{a,b \in \mathcal{A}_n} \frac{1}{2^{|a \cup b|}} + \dots.$$

Therefore it is natural to ask whether condition (c) in 2.3 can be replaced by the condition

$$\sum_{n=1}^\infty \sum_{a \in \mathcal{A}_n} \frac{1}{2^{|a|}} < \infty$$

or in general by

$$\sum_{n=1}^\infty \sum_{a \in \mathcal{A}_n} \prod_{i \in a} p_i < \infty.$$

Surprisingly the answer turns out to be negative – the following example was found by M. Talagrand ([T3]).

Theorem 2.8 (Talagrand) *Assume CH. There exists a measurable filter $\mathcal{F}$ such that for every sequence $\{J_n : n \in \omega\}$ of finite subsets of ω satisfying*

$$\sum_{n=1}^{\infty} \frac{1}{2^{|J_n|}} < \infty$$

there exists $X \in \mathcal{F}$ which contains no set J_n for $n \in \omega$.

PROOF: Let us start with the following observation:

Lemma 2.9 *Let I be a finite set of size $2n$ for some $n \in \omega$ and let λ_I be the counting measure on the set $Z(I) = \{I' \subset I : |I'| = n\}$. Suppose that C is a subset of I. Then*

$$\lambda_I(\{I' \in Z(I) : C \subset I'\}) \leq \frac{1}{2^{|C|}}.$$

PROOF: Suppose that $|C| = m$. Then the left-hand side is equal to

$$H(m) = \frac{\binom{2n-m}{n-m}}{\binom{2n}{n}} \text{ so } H(m) = \frac{n-m}{2n-m} \cdot H(m-1) \leq \frac{H(m-1)}{2}. \blacksquare$$

Lemma 2.10 *Let $\{I_n : n \in \omega\}$ be a sequence of pairwise disjoint subsets of ω each of them having even number of elements. Suppose that $\{J_n : n \in \omega\}$ is a sequence of finite subsets of ω satisfying*

$$\sum_{n=1}^{\infty} \frac{1}{2^{|J_n|}} < \infty.$$

Then for $n \in \omega$ there are sets $I'_n \subset I_n$ of size $|I_n|/2$ such that $\bigcup_{n\in\omega} I'_n$ contains no set J_n for $n \in \omega$.

PROOF: Provide $Z(I_n)$ with the counting measure λ_{I_n} and $\mathbf{P} = \prod_{n\in\omega} Z(I_n)$ with the product measure $\lambda = \prod_{n\in\omega} \lambda_{I_n}$. Using lemma 2.9 we get that for every $k \in \omega$

$$\lambda(\{\{I'_n : n \in \omega\} \in \mathbf{P} : J_k \subseteq \bigcup_{n\in\omega} I'_n\}) \leq \prod_{n\in\omega} \lambda_{I_n}(\{I'_n \in Z(I_n) : J_k \cap I_n \subseteq I'_n\}) \leq$$

$$\leq \prod_{n\in\omega} \frac{1}{2^{|I_n \cap J_k|}} = \frac{1}{2^{|J_k|}}.$$

Therefore

$$\lambda(\{\{I'_n : n \in \omega\} \in \mathbf{P} : \forall k \in \omega \ J_k \not\subseteq \bigcup_{n\in\omega} I'_n\}) \geq \prod_{k\in\omega}(1 - \frac{1}{2^{|J_k|}}) > 0.$$

In particular the set of sequences we are looking for has positive λ-measure. ■

CONSTRUCTION OF THE FILTER

Let $\{I_{k,l} : k, l \in \omega\}$ be a family of pairwise disjoint sets such that $|I_{k,l}| = 2^k$ for $k, l \in \omega$. Let $\{J_n^\xi : n \in \omega, \xi < \omega_1\}$ be an enumeration of all sequences such that

$$\sum_{n=1}^{\infty} \frac{1}{2^{|J_n^\xi|}} < 1.$$

Construct by induction a sequence $\{X_\xi : \xi < \omega_1\}$ of subsets of ω such that:

1. $J_n^\xi \not\subseteq X_\xi$ for $n \in \omega, \xi < \omega_1$,
2. family $\{X_\eta : \eta < \xi\}$ has finite intersection property for $\xi < \omega_1$,
3. for every $\xi < \omega_1$ and $\eta_1, \ldots, \eta_n < \xi$ there exists a sequence of natural numbers $\{a_k : k \in \omega\}$ such that $\lim_{k\to\infty} a_k = \infty$ and $|X_{\eta_1} \cap \cdots \cap X_{\eta_n} \cap I_{k,l}| \geq a_k$ for $l \in \omega$.

Notice that it is enough to finish the proof: let $\mathcal{F}$ be the filter generated by the family $\{X_\xi : \xi < \omega_1\}$. Clearly $\mathcal{F}$ avoids every small set $(\mathcal{A}_n)_{n=1}^\infty$ such that $\sum_{n=1}^\infty \sum_{a\in\mathcal{A}_n} 2^{-|a|} < \infty$. Moreover $\mathcal{F}$ is null since $\mathcal{F}$ is contained in the set

$$\{X \subseteq \omega : \exists k \in \omega \ \forall l \in \omega \ X \cap I_{k,l} \neq \emptyset\}$$

which is null.

Therefore assume that $\{X_\beta : \beta < \alpha < \omega_1\}$ are already constructed. Order those sets in order type ω say $\{Y'_n : n \in \omega\}$ and define

$$Y_n = Y'_1 \cap \cdots \cap Y'_n \text{ for } n \in \omega.$$

By the induction hypothesis there are sequences $\{a^n_k : k, n \in \omega\}$ such that $\lim_{k\to\infty} a^n_k = \infty$ for $n \in \omega$ and $|Y_n \cap I_{k,l}| \geq a^n_k$ for $k, l, n \in \omega$. Find a sequence $\{k_n : n \in \omega\}$ such that $\lim_{n\to\infty} a^n_{k_n} = \infty$. Let

$$X_n = Y_n \cap \bigcup_{l\in\omega} \bigcup_{j=k_n}^{k_{n+1}} I_{j,l} \text{ for } n \in \omega.$$

Let $X'_\alpha = \bigcup_{n\in\omega} X_n$. Now apply 2.10 to the sequence $\{J^\alpha_n : n \in \omega\}$ and partition $\{X'_\alpha \cap I_{k,l} : k, l \in \omega\}$ to get a sequence $\{I'_{k,l} : k, l \in \omega\}$. Let

$$X_\alpha = \bigcup_{k,l\in\omega} I'_{k,l}.$$

Verification that X_α is the element we are looking for is straightforward: clearly X_α intersects every set Y_n and avoids the sequence $\{J^\alpha_n : n \in \omega\}$. $\blacksquare$

Theorem 2.11 *Every $\mu_{\hat{p}}$-measurable filter extends to a $\mu_{\hat{p}}$-measurable filter which does not have the Baire property.*

Proof: Let $\mathcal{F}$ be a measurable filter. By 2.3 we can find a family $\{\mathcal{A}_n : n \in \omega\}$ such that $\mathcal{F} \subset (\mathcal{A}_n)^\infty_{n=1}$. For $X \subset \omega$ let $A_X = \{n \in \omega : \ \exists a \in \mathcal{A}_n \ a \subset X\}$. It is easy to see that the family $\{A_X : X \in \mathcal{F}\}$ has finite intersection property. Let $\mathcal{G}$ be any ultrafilter (or filter which does not have the Baire property) containing this family. Define

$$\mathcal{H} = \{X \subseteq \omega : A_X \in \mathcal{G}\}.$$

It is not very hard to see that the filter $\mathcal{H}$ has required properties. $\blacksquare$

3 Intersections of filters

This section is devoted to the problem of measurability of the intersection of family of filters. Let us start with countable case.

Theorem 3.1 (Talagrand [T1]) *Intersection of countably many nonmeasurable filters is a nonmeasurable filter.* $\blacksquare$

Theorem 3.2 (Talagrand [T1]) *Intersection of countable family of filters without the Baire property is a filter without the Baire property. Martin's Axiom implies that intersection of less than* $2^{\aleph_0}$ *filters without the Baire property does not have the Baire property.* $\blacksquare$

Surprisingly the second part of the above theorem does not generalize when category is replaced by measure. In fact we have the following:

Theorem 3.3 (Fremlin [F]) *Assume Martin's Axiom. Then there exists a family of Lebesgue nonmeasurable filters of cardinality* $2^{\aleph_0}$ *such that every uncountable subfamily has measurable intersection.* $\blacksquare$

The next theorem shows that the above pathology cannot happen if we assume stronger measurability properties.

Theorem 3.4 *Assume Martin's Axiom. Let* $\mu_{\hat{p}}$ *be a measure such that* $\lim_{n\to\infty} p_n = 0$ *and let* $\{\mathcal{F}_\xi : \xi < \lambda < 2^\omega\}$ *be a family of* $\mu_{\hat{p}}$*-nonmeasurable filters. Then*

$$\bigcap_{\xi<\lambda} \mathcal{F}_\xi \text{ is a Lebesgue nonmeasurable filter.}$$

PROOF: In fact we show that $\bigcap_{\xi<\lambda} \mathcal{F}_\xi$ is $\mu_{\hat{q}}$-nonmeasurable for any sequence $\hat{q}$ such that $\lim_{n\to\infty} \frac{q_n}{p_n} = \infty$. Let $\hat{q} = \{q_n : n \in \omega\}$ be a sequence satisfying the above condition and let $(\mathcal{A}_n)_{n=1}^\infty$ be any $\mu_{\hat{q}}$-small set. For given $X \subseteq \omega$ let $(\mathcal{A}_n - X)_{n=1}^\infty = (\{a - X : a \in \mathcal{A}_n\})_{n=1}^\infty$. Notice that if $X \in (\mathcal{A}_n)_{n=1}^\infty$ then $(\mathcal{A}_n - X)_{n=1}^\infty = 2^\omega$.

Define by induction sequences $\{X_\xi : \xi \leq \lambda\}$ and $\{\hat{p}^\xi : \xi \leq \lambda\} \subset \Re^\omega$ such that

1. $X_\xi \in \mathcal{F}_\eta$ for $\eta < \xi < \lambda$,

2. $X_\xi - X_\eta$ is finite for $\xi < \eta \leq \lambda$,

3. $p_n = p_n^\lambda < p_n^\xi \leq \frac{1}{2} p_n^\eta \leq q_n$ for $\eta < \xi$ and all but finitely many $n \in \omega$,

4. $\lim_{n\to\infty} \frac{p_n^\xi}{p_n} = \infty$ for $\xi < \lambda$,

5. $\mu_{\hat{p}^\xi}((\mathcal{A}_n - X_\xi)_{n=1}^\infty) = 0$ for $\xi \leq \lambda$.

It is easy to see that it is enough to finish the proof: by 1) and 2) $X_\lambda \in \bigcap_{\xi<\lambda} \mathcal{F}_\xi$ and $X_\lambda \notin (\mathcal{A}_n)_{n=1}^\infty$ by 5) and the remark above.

Suppose that $\{X_\xi : \xi < \alpha\}$ and $\{\hat{p}^\xi : \xi < \alpha\}$ are already constructed and satisfy conditions $1) - 5)$.

Case 1 $\alpha = \beta + 1$

Let $\varphi : 2^\omega \times 2^\omega \longrightarrow 2^\omega$ be defined as $\varphi(X,Y)(n) = \max\{X(n), Y(n)\}$. Notice that $\varphi(X,Y)$ is essentially the same as $X \cup Y$.

Define $p_n^\alpha = 1 - \sqrt{1 - p_n^\beta}$ for $n \in \omega$ and let ν be a measure on $2^\omega \times 2^\omega$ defined as $\mu_{\hat{p}^\alpha} \times \mu_{\hat{p}^\alpha}$. As in 1.2 we show that

$$\nu = \varphi^{-1}(\mu_{\hat{p}^\beta}).$$

Since by the induction hypothesis $\mu_{\hat{p}^\beta}((\mathcal{A}_n - X_\beta)_{n=1}^\infty) = 0$ we have that $\nu(\varphi^{-1}((\mathcal{A}_n - X_\beta)_{n=1}^\infty)) = 0$. We also have that $\mathcal{F}_\beta$ is $\mu_{\hat{p}^\alpha}$-nonmeasurable because for almost every $n \in \omega$ $p_n \leq \frac{1}{2} p_n^\beta \leq 1 - \sqrt{1 - p_n^\beta}$. Therefore by Fubini theorem there exists $X \in \mathcal{F}_\beta$ such that

$$\mu_{\hat{p}^\alpha}(\{Y \subseteq \omega : \varphi(X,Y) \in (\mathcal{A}_n - X_\beta)_{n=1}^\infty\}) = 0.$$

Let $X_\alpha = X_\beta \cup X$.

By the above remarks we have $\mu_{\hat{p}^\alpha}((\mathcal{A}_n - X_\alpha)_{n=1}^\infty)) = 0$. It is easy to check that other conditions are satisfied as well.

Case 2 α is a limit ordinal.

For this case we use Martin's Axiom: first we construct a sequence $\hat{p}^\alpha$ satisfying $3), 4)$ and 5) and then X_α satisfying 1) and 2).

Let $\mathcal{P}$ be the following notion of forcing:

$$\mathcal{P} = \{\langle s, k, H\rangle : s \in \mathbf{Q}^{<\omega}, k \in \omega, H \in [\alpha]^{<\omega} \ \&$$

$$\forall m > k \ \frac{\min\{p_m^\xi : \xi \in H\}}{p_m} > lh(s)\}.$$

($\mathbf{Q}$ is the set of rationals). For any $\langle s,k,H\rangle, \langle s',k',H'\rangle \in \mathcal{P}$ define

$$\langle s,k,H\rangle \leq \langle s',k',H'\rangle \text{ iff } s' \subseteq s \ \&\ k' \leq k \ \&\ H' \subseteq H \ \&\ \text{ for all } n \geq k'$$

$$lh(s') \cdot p_n \leq s(n) \leq \min\{p_n^\xi : \xi \in H'\}.$$

One easily checks that $\mathcal{P}$ is ccc. Define for $\xi < \alpha$ $D_\xi = \{\langle s,k,H\rangle : \xi \in H\}$. It is easy to see that these sets are dense in $\mathcal{P}$. If $\mathbf{G}$ is a filter which intersects all of them define

$$\hat{p}' = \bigcup\{s : \langle s,k,H\rangle \in \mathbf{G}\} \text{ and } p_n^\alpha = \frac{p_n'}{2} \text{ for } n \in \omega.$$

It is not very hard to check that this is a sequence we were looking for. Now we construct X_α.

By the induction hypothesis we know that $\mu_{\hat{p}^\xi}((\mathcal{A}_n - X_\xi)_{n=1}^\infty)) = 0$ for $\xi < \alpha$. Therefore by 2.5 $\mu_{\hat{p}^\alpha}((\mathcal{A}_n - X_\alpha)_{n=1}^\infty)) = 0$ for $\xi < \alpha$ which is equivalent to $\sum_{n=1}^\infty \mu_{\hat{p}^\alpha}(\{X \subset \omega : \exists a \in \mathcal{A}_n - X_\xi\ a \subset X\}) < \infty$ for $\xi < \alpha$.

Using Martin's Axiom find $\sum_{n=1}^\infty \varepsilon_n < \infty$ such that

$$\forall \xi < \alpha\ \forall^\infty n\ \mu_{\hat{p}^\alpha}(\{X \subset \omega : \exists a \in \mathcal{A}_n - X_\xi\ a \subset X\}) < \varepsilon_n.$$

Let $\mathcal{Q}$ be the following notion of forcing:

$$\mathcal{Q} = \{\{\langle n_{\alpha_1}, X_{\alpha_1}\rangle, \langle n_{\alpha_2}, X_{\alpha_2}\rangle, \ldots, \langle n_{\alpha_k}, X_{\alpha_k}\rangle\} : k \in \omega \ \&\ \alpha_i < \alpha \text{ for } i < k \ \&$$

$$\text{for all } n \in \omega\ \mu_{\hat{p}^\alpha}(\{X \subset \omega : \exists a \in \mathcal{A}_n - (\bigcup_{i<k} \langle X_{\alpha_i} - n_{\alpha_i}\rangle\ a \subset X\}) < \varepsilon_n\})\}.$$

For $p, q \in \mathcal{Q}$ define $p \leq q$ iff $p \supseteq q$.

Claim 3.5 *$\mathcal{Q}$ is ccc.*

Proof: Let $W \subseteq \mathcal{Q}$ be an uncountable family. By "thinning out" we can assume that there are $k, n_1, \ldots n_k \in \omega$ such that every element of W is of the form $\{\langle n_1, X_{\alpha_1}\rangle, \ldots, \langle n_k, X_{\alpha_k}\rangle\}$. Observe that for every $X_{\alpha_1}, \ldots, X_{\alpha_j}$ there is $n \in \omega$ such that $\{\langle n, X_{\alpha_1}\rangle, \ldots, \langle n, X_{\alpha_j}\rangle\} \in \mathcal{Q}$. This is because sets $\{X_\beta : \beta < \alpha\}$ form an increasing sequence.

Since W is an uncountable antichain we can find a number $n \in \omega$ and two conditions $\{\langle n_1, X_{\alpha_1}\rangle, \ldots, \langle n_k, X_{\alpha_k}\rangle\} \in W$ and $\{\langle n_1, X_{\beta_1}\rangle, \ldots, \langle n_k, X_{\beta_k}\rangle\} \in W$ such that $\{\langle n, X_{\alpha_1}\rangle, \ldots, \langle n, X_{\alpha_k}\rangle, \langle n, X_{\beta_1}\rangle, \ldots, \langle n, X_{\beta_k}\rangle\} \in \mathcal{Q}$ and

$$n \cap \bigcup_{i<k} \langle X_{\alpha_i} - n_{\alpha_i}\rangle = n \cap \bigcup_{i<k} \langle X_{\beta_i} - n_{\beta_i}\rangle.$$

Thus conditions $\{\langle n_1, X_{\alpha_1}\rangle, \dots, \langle n_k, X_{\alpha_k}\rangle\}$ and $\{\langle n_1, X_{\beta_1}\rangle, \dots, \langle n_k, X_{\beta_k}\rangle\}$ are compatible, which finishes the proof. ■

Let $D_\xi = \{p \in \mathcal{Q} : \exists n \in \omega \ \langle n, X_\xi\rangle \in p\}$ for $\xi < \alpha$. It is easy to see that all sets D_ξ are dense in $\mathcal{Q}$. Let $\mathbf{G}$ be a filter intersecting all D_ξ's. Define

$$X_\alpha = \bigcup \{X_\xi - n_\xi : \exists p \in \mathbf{G} \ \langle n_\xi, X_\xi\rangle \in p\}.$$

Verification that X_α satisfies conditions $1) - 5)$ is straightforward. ■

Notice that if the family of filters is countable we do not need Martin's Axiom. Using the same method we can prove Talagrand's theorem from the first section.

Corollary 3.6 (Talagrand) *Let $\{\mathcal{F}_n : n \in \omega\}$ be a countable family of $\mu_{\hat{p}}$-nonmeasurable filters. Then*

$$\bigcap_{n\in\omega} \mathcal{F}_n \text{ is a } \mu_{\hat{p}}\text{-nonmeasurable filter.}$$

Proof: Let $(\mathcal{A}_n)_{n=1}^\infty$ be any $\mu_{\hat{p}}$-small set. Construct a sequence $\{X_n : n \in \omega\}$ as in the proof of 2.4 for measures $\mu_{\hat{p}_m}$ $m \in \omega$ where $p_n^m = 2^{-m} p_n$ for $n, m \in \omega$. Set X_ω will witness that $\bigcap_{n\in\omega} \mathcal{F}_n$ is not covered by $(\mathcal{A}_n)_{n=1}^\infty$. Use the fact that $\mathcal{F}_n$ is $\mu_{\hat{p}^m}$-nonmeasurable for $n, m \in \omega$ [T1]. ■

Therefore, if we have Martins's Axiom countable case generalizes to uncountable provided we have little bit stronger measurability hypothesis.

4 Filters which are both null and meager

This section is devoted to filters which are both null and meager. Let $\mathcal{F}$ be a $\mu_{\hat{p}}$-measurable filter. By 2.3 $\mathcal{F}$ can be covered by some $\mu_{\hat{p}}$-small set $(\mathcal{A}_n)_{n=1}^\infty$. For $X \in \omega$ define $\text{supp}(X) = \{n \in \omega : \exists a \in \mathcal{A}_n \ a \subset X\}$ and let

$$\mathcal{F}^\star = \{\text{supp}(X) : X \in \mathcal{F}\}.$$

Notice that the definition of $\mathcal{F}^\star$ makes sense only in presence of some covering $(\mathcal{A}_n)_{n=1}^\infty$ of $\mathcal{F}$. It is easy to see that $\mathcal{F}^\star$ is a filter which is a continuous image of $\mathcal{F}$.

Lemma 4.1 *If $\mathcal{F}$ is $\mu_{\hat{p}}$-measurable and $\mathcal{F}^\star$ has the Baire property then $\mathcal{F}$ can be covered by a $\mu_{\hat{p}}$-null set of type F_σ.*

PROOF: Suppose that $\mathcal{F} \subseteq (\mathcal{A}_n)_{n=1}^\infty$. If $\mathcal{F}^\star$ has the Baire property then using theorem 1.4 we can find a partition of ω $\{I_n : n \in \omega\}$ such that

$$\forall X \in \mathcal{F}^\star \ \forall^\infty n \ X \cap I_n \neq \emptyset.$$

As a consequence we get

$$\mathcal{F} \subseteq \bigcup_{n\in\omega} \bigcap_{m\geq n} \bigcup_{k\in I_m} \{X \subseteq \omega : \exists a \in \mathcal{A}_k \ a \subset X\}.$$

The above set is a $\mu_{\hat{p}}$-null set of type F_σ. ■

Corollary 4.2 *Every Borel (analytic) filter can be covered by a null set of type F_σ.* ■

Notice that if $\mathcal{F}$ can be covered by a null set of type F_σ then $\mathcal{F}$ is measurable and has the Baire property. The next theorem shows that the converse does not hold. For simplicity let us work with Lebesgue measure.

Theorem 4.3 *Assume that there exists a nonmeasurable filter having the Baire property. Then there exists a filter which is both null and meager but cannot be covered by a null set of type F_σ.*

PROOF: Let us first notice that the existence of nonmeasurable filter having the Baire property follows from Martin's Axiom but it is not provable in ZFC. (see [BGJS]).

Let $\mathcal{G}$ be a nonmeasurable filter with the Baire property and $\mathcal{H}$ any filter without the Baire property.

Let $\{I_n : n \in \omega\}$ be a partition witnessing that $\mathcal{G}$ is meager. We can assume that $|I_n| > n$ for $n \in \omega$. Define

$$\mathcal{F} = \{X \in \mathcal{G} : \{n \in \omega : I_n \subset X\} \in \mathcal{H}\}.$$

It is very easy to verify that $\mathcal{F}$ is a filter, $\mathcal{F}$ is meager since $\mathcal{F} \subseteq \mathcal{G}$ and $\mathcal{F}$ is null since $\mathcal{F} \subseteq \{X \in \omega : \exists^\infty n \ I_n \subset X\}$ which is null. We will show that $\mathcal{F}$ cannot be covered by a null set of type F_σ. Let $K \subset 2^\omega$ be such a

set. First find an increasing sequence of closed sets $\{C_n : n \in \omega\}$ such that $K \subset \bigcup_{n\in\omega} C_n$ and $\mu(C_n) = 0$ for $n \in \omega$. Now for $n, m \in \omega$ define

$$C_m^n = \{s \in 2^m : [s] \cap C_n \neq \emptyset\}.$$

Let $\{k_n : n \in \omega\}$ be any sequence of natural numbers such that

$$\sum_{n=1}^{\infty} 2^{k_n} \cdot \mu(V(C_{k_{n+1}}^n)) < \infty.$$

Define for $n \in \omega$

$$U_n = [k_n, k_{n+1}) \text{ and}$$
$$T_n = \{s \in 2^{U_n} : \exists t \in C_{k_{n+1}}^n \; s \restriction U_n = t \restriction U_n\}.$$

From the above definitions easily follows that

$$K \subseteq \{x \in 2^\omega : \forall^\infty n \; x \restriction U_n \in T_n\} \subseteq (U_n, T_n)_{n=1}^\infty$$

and that the set $(U_n, T_n)_{n=1}^\infty$ is small. Without loss of generality we can also assume that

$$\forall n \in \omega \; \exists m \in \omega \; I_n \subset U_m.$$

Since $\mathcal{G}$ is a nonmeasurable filter we can find $X \in \mathcal{G} - (U_n, T_n)_{n=1}^\infty$. Using theorem 1.4 and the fact that $\mathcal{H}$ does not have the Baire property we can also find an element $Y \in \mathcal{H}$ such that for some infinite set $S \subseteq \omega$

$$(\star) \quad \bigcup_{n\in Y} I_n \cap \bigcup_{n\in S} U_n = \emptyset.$$

Let

$$Z = X \cup \bigcup_{n\in Y} I_n.$$

We will show that $Z \in \mathcal{F} - K$ which finishes the proof since K is an arbitrary F_σ set. Obviously $Z \in \mathcal{F}$ and $Z \notin \{x \in 2^\omega : \forall^\infty n \; x \restriction U_n \in T_n\}$ because of $(\star)$ and the definition of X. ■

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